Mass dependence of the vacuum energy density in the massive Schwinger model

Taekoon Lee*

Department of Physics, Kunsan National University, Kunsan 573-701, Korea

The vacuum energy density of the massive Schwinger model is shown to be not power expandable in the fermion mass.

The exact operator solution of the massless Schwinger model [1] shows that the Hilbert space of the Schwinger model is given as the product of that of a free, massive scalar field and the theta vacua [2, 3, 4]. The theta vacua are known to arise from the anomalous breakdown of the chiral symmetry and the nontrivial topology of the field configurations in two-dimensional $U(1)$ gauge theory. The vacua may be labeled by the phase of the chiral condensate.

Although the physics of the Schwinger model should not depend on any particular choice of the vacuum among the theta vacua one could ask whether it is at all possible to choose a vacuum in the Schwinger model. In a theory with the ordinary spontaneous symmetry breaking a particular vacuum may be picked up out of the continuum of vacua by adding an infinitesimal symmetry breaking term in such a way that the energy shift due to the added term be minimized at the selected vacuum. This is the vacuum alignment [5].

In the case of the theta vacua it was argued that the vacuum alignment is not possible owing to the absence of the Goldstone boson associated with the anomalous chiral symmetry breaking [6]. The implication of this is that the phase of the chiral condensate in the Schwinger model cannot be picked up by adding an infinitesimal fermion mass term to the Schwinger model. Instead, it is widely accepted that a theta vacuum, and the corresponding phase in the chiral condensate, can be selected by adding a topological term to the Schwinger model. This is peculiar, however, since it would suggest that the original Schwinger model have a unique vacuum which, if true, would contradict the operator solution.

A consequence of the above uniqueness of the vacuum in the Schwinger model is the power expandability of the vacuum energy of the massive Schwinger model in the fermion mass [6, 7]. To be specific, consider the Lagrangian density of the massive Schwinger model

$$\mathcal{L}_{\rm A} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}(\partial\!\!\!/ + ie\, A\!\!\!/)\psi + |m|(e^{i\alpha}\bar{\psi}_{\rm L}\psi_{\rm R} + \text{h.c.})\,, \tag{1}$$

where $\psi_{\rm L,R} = \frac{1}{2}(1\pm\gamma_5)\psi$ and α denotes the phase of the fermion mass $m = |m|e^{i\alpha}$.

Usually, invoking parity symmetry, the vacuum of the massless Schwinger model is assumed to have a real chiral condensate, which allows one to write the chiral condensate in the vacuum of the model (1) as

$$<\bar{\psi}_{\rm L}\psi_{\rm R}>_{\rm A} = \Sigma + O(m), \tag{2}$$

where Σ is a real parameter and $< >_{\rm A}$ denotes the expectation values in the vacuum of (1). Then using the mass perturbation the vacuum energy density of (1) can be written as

$$\varepsilon_{\rm A} = \varepsilon_0 - \Sigma(m + m^*) + O(m^2)\,, \tag{3}$$

where ε_0 denotes the vacuum energy density of the massless Schwinger model. The vacuum energy density thus can be expanded in powers of the fermion mass and its complex conjugate.

On the other hand, if the theta vacua are alignable by the fermion mass term, the chiral condensate would get a phase that cancels the fermion mass phase exactly in order to minimize the energy shift by the mass term. The vacuum energy density then would depend on $|m|$ only, and it could not be expanded in powers of the fermion mass. Therefore, whether the theta vacua are alignable or not can be determined by looking at the mass dependence of the vacuum energy density.

In this note we give a simple argument that shows the vacuum energy density (3) cannot be compatible with the anomaly equation for the chiral symmetry, and show that the vacuum energy density that is consistent with the anomaly equation depends only on the magnitude $|m|$ of the fermion mass.

To show this we consider the following chirally rotated form of the Lagrangian density (1) [8]

$$\mathcal{L}_{\rm B} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}(\partial\!\!\!/ + ie\, A\!\!\!/)\psi + |m|(e^{i(\alpha-\beta)}\bar{\psi}_{\rm L}\psi_{\rm R} + \text{h.c.}) + \frac{\beta}{2}\tilde{F}, \tag{4}$$

*tlee@kunsan.ac.kr

where β is a real parameter and $\tilde{F}$ denotes the topological term given by

$$\tilde{F} = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}\,, \tag{5}$$

where $\epsilon^{\mu\nu}$ is an antisymmetric tensor with $\epsilon^{01}=1$. The difference $\delta\mathcal{H}$ of the Hamiltonian densities of the model (4) and the massless Schwinger model is given, up to an irrelevant, field-independent constant term, by

$$\delta\mathcal{H} = -|m|(e^{i(\alpha-\beta)}\bar{\psi}_{\rm L}\psi_{\rm R} + {\rm h.c.}) - \frac{\beta}{2}\tilde{F}\,, \tag{6}$$

and the vacuum energy density of (4) then by

$$\varepsilon_{\rm B} = \varepsilon_0 - 2|m|{\rm Re}(e^{i(\alpha-\beta)} <\bar{\psi}_{\rm L}\psi_{\rm R}>_{\rm B}) - \frac{\beta}{2} <\tilde{F}>_{\rm B} + O(m^2)\,, \tag{7}$$

where $<\ >_{\rm B}$ denotes the expectation values in the vacuum of (4).

Now suppose the condensate (2) is correct. Then the chiral condensate of the model (4) will be given as

$$<\bar{\psi}_{\rm L}\psi_{\rm R}>_{\rm B} = \Sigma e^{i\beta} + O(m)\,, \tag{8}$$

since the condensates of the two vacua are related by $<\bar{\psi}_{\rm L}\psi_{\rm R}>_{\rm A} = <\bar{\psi}_{\rm L}\psi_{\rm R}>_{\rm B} e^{-i\beta}$. Using the anomaly equation for the chiral current $J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ the expectation value of the topological charge density can be related to the chiral condensate by

$$0 = \partial\mu <J_5^\mu>_{\rm B} = 2i|m|\big(e^{i(\alpha-\beta)} <\bar{\psi}_{\rm L}\psi_{\rm R}>_{\rm B} - e^{-i(\alpha-\beta)} <\bar{\psi}_{\rm R}\psi_{\rm L}>_{\rm B}\big) - <\tilde{F}>_{\rm B}\,, \tag{9}$$

which gives

$$<\tilde{F}>_{\rm B} = -4|m|\Sigma\sin\alpha + O(m^2)\,. \tag{10}$$

Substituting (8) and (10) into (7) we get

$$\varepsilon_{\rm B} = \varepsilon_0 - 2|m|\Sigma\cos\alpha + 2|m|\Sigma\beta\sin\alpha + O(m^2)\,. \tag{11}$$

Since the Lagrangian densities (1) and (4) are equivalent $\varepsilon_{\rm A}$ and $\varepsilon_{\rm B}$ must have the same m dependence for any β but obviously this is impossible with (11). We, therefore, conclude that the widely accepted expressions (2) and (3) cannot be compatible with the anomaly equation.

To find the chiral condensate that is consistent with the anomaly equation we modify the condensate (2) by adding a phase factor as

$$<\bar{\psi}_{\rm L}\psi_{\rm R}>_{\rm A} = \Sigma e^{i\chi} + O(m)\,, \tag{12}$$

where Σ is now assumed to be a positive real parameter. The chiral condensate of (4) corresponding to this is then given by

$$<\bar{\psi}_{\rm L}\psi_{\rm R}>_{\rm B} = \Sigma e^{i(\chi+\beta)} + O(m)\,, \tag{13}$$

and the expectation value for the topological charge density by

$$<\tilde{F}>_{\rm B} = -4|m|\Sigma\sin(\alpha+\chi)\,. \tag{14}$$

The vacuum energy density of (1) will then be given by

$$\varepsilon_{\rm A} = \varepsilon_0 - 2|m|\Sigma\cos(\alpha+\chi) + O(m^2)\,, \tag{15}$$

and that of (4) by

$$\varepsilon_{\rm B} = \varepsilon_0 - 2|m|\Sigma\cos(\alpha+\chi) + 2|m|\Sigma\beta\sin(\alpha+\chi) + O(m^2)\,. \tag{16}$$

For these vacuum energy densities to have the identical m dependence for all β the phase χ must satisfy

$$\sin(\alpha+\chi) = 0\,, \tag{17}$$

which shows the phase of the chiral condensate is governed by that of the fermion mass. This means that the theta vacua must be alignable by the fermion mass term!

Note that Eq. (17) shows the vacuum expectation value for the topological charge density always vanishes and the vacuum energy density does not depend on the phase of the fermion mass. The implication of this is that the true vacuum of the massive Schwinger model is CP symmetric irrespective of the phase of the fermion mass or the presence of the topological term. Since the energy shift by the mass term is minimized at $\chi = -\alpha$ the vacuum energy density is given by

$$\varepsilon_{\rm A} = \varepsilon_{\rm B} = \varepsilon_0 - 2|m\Sigma| + O(m^2)\,. \tag{18}$$

Finally, we note that the argument so far for the theta vacuum alignment is applicable to QCD as well, and that the theta vacuum alignment in QCD removes, entirely, the strong CP phase of the quark mass matrix from the QCD low energy physics [9].

Acknowledgments

This work was supported by the Korea Research Foundation Grant (KRF-2006-015-C00251).

[1] J. S. Schwinger, Phys. Rev. **128**, 2425 (1962).
[2] J. H. Lowenstein and J. A. Swieca, Ann. Phys. **68**, 172 (1971).
[3] A. K. Raina and G. Wanders, Ann. Phys. **132**, 404 (1981).
[4] G. Morchio, D. Pierotti, and F. Strocchi, Ann. Phys. **188**, 217 (1988).
[5] R. F. Dashen, Phys. Rev. **D3**, 1879 (1971).
[6] S. R. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. **93**, 267 (1975).
[7] H. Leutwyler and A. Smilga, Phys. Rev. **D46**, 5607 (1992).
[8] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979).
[9] T. Lee, Int. J. Mod. Phys. **A16**, 4321 (2001), hep-ph/0006349.